Multistable Lévy motions and their continuous approximations

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Abstract

Multistable Lévy motions are extensions of Lévy motions where the stability index is allowed to vary in time. Several constructions of these processes have been introduced recently, based on Poisson and Ferguson-Klass-LePage series representations and on multistable measures. In this work, we prove a functional central limit theorem for the independent-increments multistable Lévy motion, as well as of integrals with respect to these processes, using weighted sums of independent random variables. This allows us to construct continuous approximations of multistable Lévy motions. In particular, we prove that multistable Lévy motions are stochastic Hölder continuous and strongly localisable.

Keywords: (strong) localisability; multistable process; stochastic Hölder continuous; stable process; continuous approximation.

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1. Introduction

Recall that a stochastic process $\{L(t), t \geq 0\}$ is called (standard) $\alpha-$stable Lévy motion if the following three conditions hold:
(C1) $L(0) = 0$ almost surely;
(C2) L has independent increments;
(C3) $L(t) - L(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t$ and for some $0 < \alpha \leq 2, -1 \leq \beta \leq 1$.
Here $S_\alpha(\sigma, \beta, 0)$ stands for a stable random variable with index of stability α, scale parameter σ, skewness parameter β and shift parameter equal to 0. Recall that α governs the intensity of jumps.

Such processes have stationary increments, and they are $1/\alpha-$self-similar, that is, for all $c > 0$, the processes $\{L(c\,t), t \geq 0\}$ and $\{c^{1/\alpha} L(t), t \geq 0\}$ have the same finite-dimensional distributions. An $\alpha-$stable Lévy motion is symmetric when $\beta = 0$. Stable Lévy motions, and, more generally, stable processes have been the subject of intense activity in recent years, both on

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the theoretical side (see, e.g. [13]) and in applications [12]. However, the stationary property of their increments restricts their use in some situations, and generalizations are needed for instance to model real-world phenomena such as financial records, epileptic episodes in EEG or internet traffic. A significant feature in these cases is that the "local intensity of jumps" varies with time t. A way to deal with such a variation is set up a class of processes whose stability index α is a function of t. More precisely, one aims at defining non-stationary increments processes which are, at each time t, "tangent" (in a certain sense explained below) to a stable process with stability index $\alpha(t)$.

Formally, one says that a stochastic process $\{X(t), t \in [0,1]\}$ is *multistable* [7] if, for almost all $t \in [0,1)$, X is *localisable* at t with tangent process X'_t an $\alpha(t)-$stable process. Recall that $\{X(t), t \in [0,1]\}$ is said to be $h-$localisable at t (cf. [3, 4]), with $h > 0$, if there exists a non-trivial process X'_t, called the tangent process of X at t, such that

$$\lim_{r \searrow 0} \frac{X(t+ru) - X(t)}{r^h} = X'_t(u), \tag{1}$$

where convergence is in finite dimensional distributions.

Let $D[0,1]$ be the set of càdlàg functions on $[0,1]$, that is functions which are continuous on the right and have left limits at all $t \in [0,1]$, endowed with the Skorohod metric d_S [2]. If X and X'_t have versions in $D[0,1]$ and convergence in (1) is in distribution with respect to d_S, one says that X is $h-$*strongly localisable* at t with strong local form X'_t.

In this work, we will be concerned with the simplest non-trivial multistable processes, namely multistable Lévy motions (MsLM), which are non-stationary increments extensions of stable Lévy motions. Two such extensions exist [7, 8]:

1. The *field-based* MsLM admit the following series representation:

$$L_F(t) = C_{\alpha(t)}^{1/\alpha(t)} \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} \mathbf{1}_{[0,t]}(\mathsf{X}) \mathsf{Y}^{<-1/\alpha(t)>} \qquad (t \in [0,T]), \tag{2}$$

where Π is a Poisson point process on $[0,1] \times \mathbb{R}$ with mean measure the Lebesgue measure $\mathcal{L}$, $a^{<b>} := \text{sign}(a)|a|^b$ and

$$C_u = \left(\int_0^\infty x^{-u} \sin(x) dx \right)^{-1}. \tag{3}$$

Their joint characteristic function reads:

$$\mathbb{E} \exp \left\{ i \sum_{j=1}^m \theta_j L_F(t_j) \right\} = \exp \left\{ -2 \int_{[0,T]} \int_0^{+\infty} \sin^2 \left(\sum_{j=1}^m \theta_j \frac{C_{\alpha(t_j)}^{1/\alpha(t_j)}}{2y^{1/\alpha(t_j)}} \mathbf{1}_{[0,t_j]}(x) \right) dy\, dx \right\} \tag{4}$$

for $d \in \mathbb{N}, (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ and $(t_1, \ldots, t_d) \in \mathbb{R}^d$. These processes have correlated increments, and they are localisable as soon as the function α is Hölder-continuous.

2. The *independent-increments* MsLM admit the following series representation:

$$L_I(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} C_{\alpha(\mathsf{X})}^{1/\alpha(\mathsf{X})} \mathbf{1}_{[0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(\mathsf{X})>} \qquad (t \in [0,T]). \tag{5}$$

As their name indicates, they have independent increments, and their joint characteristic function reads:

$$\mathbb{E}\exp\left\{ i\sum_{j=1}^{d}\theta_j L_I(t_j)\right\} = \exp\left\{ -\int \Big| \sum_{j=1}^{d}\theta_j \mathbf{1}_{[0,\ t_j]}(s)\Big|^{\alpha(s)} ds \right\}, \tag{6}$$

for $d \in \mathbb{N}, (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$ and $(t_1, \dots, t_d) \in \mathbb{R}^d$. These processes are localisable as soon as the function α verifies:

$$\Big(\alpha(x) - \alpha(x+t)\Big) \ln t \to 0 \tag{7}$$

uniformly for all x in finite interval as $t \searrow 0$ [8].

Of course, when $\alpha(t)$ is a constant α for all t, both L_F and L_I are simply the Poisson representation of $\alpha-$stable Lévy motion, that we denote by L_α. In general, L_F and L_I are semi-martingales [11]. For more properties of L_F, such as Ferguson-Klass-LePage series representations and Hölder exponents, we refer to [6, 9, 10].

In this paper, we prove a functional central limit theorem for independent-increments MsLM: we show that certain weighted sums of independent random variables converge in $(D[0,1], d_S)$ to L_I. This allows us to obtain strong localisability of these processes. Moreover, we establish continuous approximations of MsLM and an alternative representation for the integrals of multistable Lévy measure. Some properties of the integrals of multistable Lévy measure are investigated. In particular, we prove that MsLM are stochastic Hölder continuous and strongly localisable.

The paper is organized as follows. In Section 2, we present the functional central limit theorem for independent-increments MsLM. In Section 3, we establish continuous approximations of MsLM. In the last section, we give a representation of MsLM and investigate some properties, including stochastic Hölder continuous and strongly localisable, of the integrals of multistable Lévy measure.

2. Functional Central Limit Theorems for Multistable Lévy Motions

We show in this section how to approximate the independent-increments MsLM in law by weighted sums of independent random variables.

Theorem 2.1. *Let $(\alpha_n(u))_n, \alpha(u), u \in [0,1]$, be a class of càdlàg functions ranging in $[a,b] \subset (0,2]$ such that the sequence $(\alpha)_n$ tends to α in the uniform metric. Let $(X(k,n))_{n\in\mathbb{N},\ k=1,\dots,2^n}$ be a family of independent and symmetric $\alpha_n(\frac{k}{2^n})-$stable random variables with unit scale parameter, i.e., $X(k,n) \sim S_{\alpha_n(\frac{k}{2^n})}(1,0,0)$. Then*

- *the sequence of processes*

$$L_I^{(n)}(u) = \sum_{k=1}^{\lfloor 2^n u \rfloor} \left(\frac{1}{2^n}\right)^{1/\alpha_n(\frac{k}{2^n})} X(k,n), \qquad u \in [0,1], \tag{8}$$

tends in distribution to $L_I(u)$ in $(D[0,1], d_S)$, where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. In particular, if α satisfies condition (7), then $L_I(u)$ is localisable at all times.

- *the sequence of processes*

$$L_R^{(n)}(u) = \sum_{k=1}^{\Gamma_{\lfloor 2^n u \rfloor}} \left(\frac{1}{2^n}\right)^{1/\alpha_n(\frac{k}{2^n})} X(k,n), \qquad u \in [0,1], \tag{9}$$

tends in distribution to $L_I(u)$ in $(D[0,1], d_S)$, where $(\Gamma_i)_{i\geq 1}$ is a sequence of arrival times of a Poisson process with unit arrival rate and is independent of $(X(k,n))_{n\in\mathbb{N},\ k=1,\ldots,2^n}$.

- *the sequence of processes*

$$L_C^{(n)}(u) = \sum_{k=1}^{\lfloor 2^n u \rfloor} \left(\frac{1}{\Gamma_{2^n}}\right)^{1/\alpha_n(\frac{k}{2^n})} X(k,n), \qquad u \in [0,1], \tag{10}$$

tends in distribution to $L_I(u)$ in $(D[0,1], d_S)$.

Proof. We prove the first claim by the following three steps.

First, we prove that $L_I^{(n)}(u)$ converges to $L_I(u)$ in finite dimensional distribution. For any $u_1, u_2 \in [0,1]$ and $u_2 > u_1$, we have, for any $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}e^{i\theta\left(L_I^{(n)}(u_2) - L_I^{(n)}(u_1)\right)} = \lim_{n\to\infty} \exp\left\{ - \sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u_2 \rfloor} \frac{1}{2^n} |\theta|^{\alpha_n(\frac{k}{2^n})} \right\}. \tag{11}$$

Notice that

$$\begin{aligned} \sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u_2 \rfloor} \frac{1}{2^n} \Big| |\theta|^{\alpha_n(\frac{k}{2^n})} - |\theta|^{\alpha(\frac{k}{2^n})} \Big| &\leq |\theta|^\tau \log|\theta| \sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u_2 \rfloor} \frac{1}{2^n} \Big| \alpha_n\Big(\frac{k}{2^n}\Big) - \alpha\Big(\frac{k}{2^n}\Big) \Big| \\ &\leq \Big|\Big| \alpha_n(\cdot) - \alpha(\cdot) \Big|\Big|_\infty |\theta|^\tau \log|\theta| \frac{\lfloor 2^n u_2 \rfloor - \lfloor 2^n u_1 \rfloor}{2^n}, \end{aligned} \tag{12}$$

where $\tau = a\mathbf{1}_{[0,1)}(|\theta|) + b\mathbf{1}_{[1,\infty)}(|\theta|)$. By hypothesis, we have

$$\lim_{n\to\infty} ||\alpha_n(\cdot) - \alpha(\cdot)||_\infty = 0.$$

Thus inequality (12) implies that

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u_2\rfloor}\frac{1}{2^n}|\theta|^{\alpha_n(\frac{k}{2^n})} &= \lim_{n\to\infty}\sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u_2\rfloor}\frac{1}{2^n}|\theta|^{\alpha(\frac{k}{2^n})}\\ &= \int_{u_1}^{u_2}|\theta|^{\alpha(s)}ds.\end{aligned}$$

From (11), it follows that

$$\lim_{n\to\infty}\mathbb{E}e^{i\theta\left(L_I^{(n)}(u_2)-L_I^{(n)}(u_1)\right)} = \exp\left\{-\int_{u_1}^{u_2}|\theta|^{\alpha(s)}ds\right\}. \tag{13}$$

Hence $L_I^{(n)}(u_2)-L_I^{(n)}(u_1)$ converges in distribution and the characteristic function of its limit is defined by (13). Since $L_I^{(n)}(u)$ has independent increments, the limit of $L_I^{(n)}(u)$ has the joint characteristic function (6), i.e., $L_I^{(n)}(u)$ converges to $L_I(u)$ in finite dimensional distribution.

Second, we prove that $L_I^{(n)}(u)$ converges to $L_I(u)$ in $(D[0,1],d_S)$. By Theorem 15.6 of Billingsley [2], it suffices to show that

$$\mathbb{P}\Big(\Big|L_I^{(n)}(u)-L_I^{(n)}(u_1)\Big|\geq\lambda,\ \Big|L_I^{(n)}(u_2)-L_I^{(n)}(u)\Big|\geq\lambda\Big)\leq\frac{C}{\lambda^{2\gamma}}\Big[u_2-u_1\Big]^2 \tag{14}$$

for $u_1\leq u\leq u_2,\lambda>0$ and $n\geq 1$, where $\gamma=a\mathbf{1}_{[2,\infty)}(\lambda)+b\mathbf{1}_{(0,2)}(\lambda)$ and C is a constant depending only on a and b. If $u_2-u_1<1/2^n$, then either $L_I^{(n)}(u_2)=L_I^{(n)}(u)$ or $L_I^{(n)}(u)=L_I^{(n)}(u_1)$; in either of these cases the left side of (14) vanished. Next, we consider the case of $u_2-u_1\geq 1/2^n$. Since $L_I^{(n)}(u)-L_I^{(n)}(u_1)$ and $L_I^{(n)}(u_2)-L_I^{(n)}(u)$ are independent, it follows that

$$\begin{aligned}&\mathbb{P}\Big(\Big|L_I^{(n)}(u)-L_I^{(n)}(u_1)\Big|\geq\lambda,\ \Big|L_I^{(n)}(u_2)-L_I^{(n)}(u)\Big|\geq\lambda\Big)=\\ &\qquad\mathbb{P}\Big(\Big|L_I^{(n)}(u)-L_I^{(n)}(u_1)\Big|\geq\lambda\Big)\,\mathbb{P}\Big(\Big|L_I^{(n)}(u_2)-L_I^{(n)}(u)\Big|\geq\lambda\Big).\end{aligned}$$

Then, by the Billingsley inequality (cf. p. 47 of [2]), it is easy to see that

$$\begin{aligned}\mathbb{P}\Big(\Big|L_I^{(n)}(u)-L_I^{(n)}(u_1)\Big|\geq\lambda\Big) &\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(1-\mathbb{E}e^{i\theta\left(L_I^{(n)}(u)-L_I^{(n)}(u_1)\right)}\Big)d\theta\\ &= \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(1-\exp\Big\{-\sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u\rfloor}\frac{1}{2^n}|\theta|^{\alpha_n(\frac{k}{2^n})}\Big\}\Big)d\theta\\ &\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u\rfloor}\frac{1}{2^n}|\theta|^{\alpha_n(\frac{k}{2^n})}d\theta\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u\rfloor} \frac{1}{2^n}\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big|\theta\Big|^{\gamma}\,d\theta \\
&\leq \frac{C_1}{\lambda^{\gamma}}\Bigg[\frac{\lfloor 2^n u\rfloor-\lfloor 2^n u_1\rfloor}{2^n}\Bigg],
\end{aligned}
$$

where C_1 is a constant depending only on a and b. Similarly, it holds

$$
\mathbb{P}\Big(\Big|L_I^{(n)}(u_2)-L_I^{(n)}(u)\Big|\geq\lambda\Big)\leq\frac{C_2}{\lambda^{\gamma}}\Bigg[\frac{\lfloor 2^n u_2\rfloor-\lfloor 2^n u\rfloor}{2^n}\Bigg], \tag{15}
$$

where C_2 is a constant depending only on a and b. Using the inequality $xy\leq(x+y)^2/4$ for all $x,y\geq 0$, we deduce

$$
\begin{aligned}
&\mathbb{P}\Big(\Big|L_I^{(n)}(u)-L_I^{(n)}(u_1)\Big|\geq\lambda,\ \Big|L_I^{(n)}(u_2)-L_I^{(n)}(u)\Big|\geq\lambda\Big) \\
\leq\ &\frac{C_1C_2}{\lambda^{2\gamma}}\Bigg[\frac{\lfloor 2^n u\rfloor-\lfloor 2^n u_1\rfloor}{2^n}\Bigg]\Bigg[\frac{\lfloor 2^n u_2\rfloor-\lfloor 2^n u\rfloor}{2^n}\Bigg] \\
\leq\ &\frac{C_1C_2}{4}\frac{1}{\lambda^{2\gamma}}\Bigg[\frac{\lfloor 2^n u_2\rfloor-\lfloor 2^n u_1\rfloor}{2^n}\Bigg]^2 \\
\leq\ &C_1C_2\frac{1}{\lambda^{2\gamma}}\Big[u_2-u_1\Big]^2,
\end{aligned}
$$

where the last line follows from the fact that

$$
\frac{\lfloor 2^n u_2\rfloor-\lfloor 2^n u_1\rfloor}{2^n}\ \leq\ \frac{2^n u_2-2^n u_1+1}{2^n}\ \leq\ 2\Big[u_2-u_1\Big].
$$

This completes the proof of (14).

Third, we prove that if α satisfies condition (7), then $L_I(u)$ is localisable at all times. Falconer and Liu (cf. Theorem 2.7 of [8]) have proved that the process $L_I(u)$, defined by the joint characteristic function (6), is localisable at u to Lévy motions $L_{\alpha(u)}(\cdot)$ with the stability index $\alpha(u)$. Here we give another proof to complete our argument. For any $(t_1,...,t_d)\in[0,1]^d$, from equality (6), it is easy to see that

$$
\begin{aligned}
&\mathbb{E}\exp\Bigg\{i\sum_{j=1}^{d}\theta_j\left(\frac{L_I(u+rt_j)-L_I(u)}{r^{1/\alpha(u)}}\right)\Bigg\} \\
&\qquad\qquad = \ \exp\Bigg\{-\int\Big|\sum_{j=1}^{d}\theta_j r^{-1/\alpha(u)}\mathbf{1}_{[u,\ u+rt_j]}(s)\Big|^{\alpha(s)}ds\Bigg\}.
\end{aligned}
$$

Setting $s=u+rt$, we find that

$$
\mathbb{E}\exp\Bigg\{i\sum_{j=1}^{d}\theta_j\left(\frac{L_I(u+rt_j)-L_I(u)}{r^{1/\alpha(u)}}\right)\Bigg\}
$$

$$= \exp\left\{ -\int \Big| \sum_{j=1}^{d} \theta_j \mathbf{1}_{[0,\ t_j]}(t) \Big|^{\alpha(u+rt)} r^{(\alpha(u)-\alpha(u+rt))/\alpha(u)} dt \right\}.$$

By condition (7), it follows that

$$\lim_{r\searrow 0} r^{(\alpha(u)-\alpha(u+rt))/\alpha(u)} = 1 \qquad \text{and} \qquad \lim_{r\searrow 0} \alpha(u+rt) = \alpha(u). \tag{16}$$

Hence, using dominated convergence theorem, we have

$$\begin{aligned} \lim_{r\searrow 0} \mathbb{E}\exp\left\{ i\sum_{j=1}^{d} \theta_j \left(\frac{L_I(u+rt_j) - L_I(u)}{r^{1/\alpha(u)}} \right) \right\} &= \exp\left\{ -\int \Big| \sum_{j=1}^{d} \theta_j \mathbf{1}_{[0,\ t_j]}(t) \Big|^{\alpha(u)} dt \right\} \\ &= \mathbb{E}\exp\left\{ i\sum_{j=1}^{d} \theta_j L_{\alpha(u)}(t_j) \right\}, \end{aligned}$$

which means that $L_I(u)$ is localisable at u to an $\alpha(u)-$stable Lévy motion $L_{\alpha(u)}(t)$. This completes the proof of the first claim of the theorem.

Next, we prove the second claim of the theorem. For any $u_1, u_2 \in [0,1]$ and $u_2 > u_1$, it is easy to see that, for any $\theta \in \mathbb{R}$,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}e^{i\theta(L_R^{(n)}(u_2) - L_R^{(n)}(u_1))} &= \lim_{n\to\infty} \mathbb{E}\exp\left\{ -\sum_{k=\Gamma_{\lfloor 2^n u_1\rfloor}+1}^{\Gamma_{\lfloor 2^n u_2\rfloor}} \frac{1}{2^n} |\theta|^{\alpha_n(\frac{k}{2^n})} \right\} \\ &= \exp\left\{ -\int_{u_1}^{u_2} |\theta|^{\alpha(s)} ds \right\}, \end{aligned} \tag{17}$$

where the last line follows from the weak law of large numbers. Notice that $L_R^{(n)}(u)$ also has independent increments. The rest of the proof of the second claim is similar to the proof of the first one. For this reason, we shall not carry it out.

In the sequel, we prove the third claim by the following two steps.

First, we prove that $L_C^{(n)}(u)$ converges to $L_I(u)$ in finite dimensional distribution. It is worth noting that $L_C^{(n)}$ does not have independent increments. This property implies that we cannot use the previous method. For any $(u_1, ..., u_d) \in [0,1]^d$ and any $(\theta_1, ..., \theta_d) \in \mathbb{R}^d$ such that $0 = u_0 \le u_1 \le u_2 \le ... \le u_d$, we have

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}e^{i\sum_{j=1}^{d}\theta_j L_C^{(n)}(u_j)} &= \lim_{n\to\infty} \mathbb{E}\exp\left\{ i\sum_{l=1}^{d} \sum_{k=\lfloor 2^n u_{l-1}\rfloor}^{\lfloor 2^n u_l\rfloor} \sum_{j=l}^{d} \theta_j \Big(\frac{1}{\Gamma_{2^n}} \Big)^{1/\alpha_n(\frac{k}{2^n})} X(k,n) \right\} \\ &= \lim_{n\to\infty} \mathbb{E}\exp\left\{ -\sum_{l=1}^{d} \sum_{k=\lfloor 2^n u_{l-1}\rfloor}^{\lfloor 2^n u_l\rfloor} \Big| \sum_{j=l}^{d} \theta_j \Big|^{\alpha_n(\frac{k}{2^n})} \frac{1}{2^n} \frac{2^n}{\Gamma_{2^n}} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \exp\Big\{-\sum_{l=1}^{d}\int_{u_{l-1}}^{u_l}\Big|\sum_{j=l}^{d}\theta_j\Big|^{\alpha(s)}ds\Big\} \\
&= \exp\Big\{-\int\Big|\sum_{j=1}^{d}\theta_j\mathbf{1}_{[0,u_j)}(s)\Big|^{\alpha(s)}ds\Big\},
\end{aligned}
$$

which gives the joint characteristic function of L_I.

Second, we prove that $L_C^{(n)}(u)$ converges to $L_I(u)$ in $(D[0,1], d_S)$. Again by Theorem 15.6 of Billingsley [2], it suffices to show that

$$
\mathbb{P}\Big(\Big|L_C^{(n)}(u)-L_C^{(n)}(u_1)\Big|\geq\lambda,\ \Big|L_C^{(n)}(u_2)-L_C^{(n)}(u)\Big|\geq\lambda\Big)\leq\frac{C}{\lambda^{2\gamma}}\Big[u_2-u_1\Big]^2 \tag{18}
$$

for $u_1\leq u\leq u_2,\lambda>0$ and $n\geq 1$, where $\gamma=a\mathbf{1}_{[2,\infty)}(\lambda)+b\mathbf{1}_{(0,2)}(\lambda)$ and C is a constant depending only on a and b. We need only consider the case of $u_2-u_1\geq 1/2^n$. Since $L_C^{(n)}(u)-L_C^{(n)}(u_1)$ and $L_C^{(n)}(u_2)-L_C^{(n)}(u)$ are conditionally independent given Γ_{2^n}, it follows that

$$
\begin{aligned}
&\mathbb{P}\Big(\Big|L_C^{(n)}(u)-L_C^{(n)}(u_1)\Big|\geq\lambda,\ \Big|L_C^{(n)}(u_2)-L_C^{(n)}(u)\Big|\geq\lambda\ \Big|\ \Gamma_{2^n}\Big)= \\
&\qquad\mathbb{P}\Big(\Big|L_C^{(n)}(u)-L_C^{(n)}(u_1)\Big|\geq\lambda\ \Big|\ \Gamma_{2^n}\Big)\ \mathbb{P}\Big(\Big|L_C^{(n)}(u_2)-L_C^{(n)}(u)\Big|\geq\lambda\ \Big|\ \Gamma_{2^n}\Big).
\end{aligned}\tag{19}
$$

It is easy to see that

$$
\begin{aligned}
\mathbb{P}\Big(\Big|L_C^{(n)}(u)-L_C^{(n)}(u_1)\Big|\geq\lambda\ \Big|\ \Gamma_{2^n}\Big) &\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(1-\mathbb{E}\Big[e^{i\theta\big(L_C^{(n)}(u)-L_C^{(n)}(u_1)\big)}\ \Big|\ \Gamma_{2^n}\Big]\Big)d\theta \\
&= \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(1-\mathbb{E}\Big[\exp\Big\{-\sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u\rfloor}\frac{1}{\Gamma_{2^n}}|\theta|^{\alpha_n(\frac{k}{2^n})}\Big\}\ \Big|\ \Gamma_{2^n}\Big]\Big)d\theta \\
&\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\sum_{k=\lfloor 2^n u_1\rfloor+1}^{\lfloor 2^n u\rfloor}\frac{1}{\Gamma_{2^n}}|\theta|^{\alpha_n(\frac{k}{2^n})}d\theta \\
&\leq \frac{C_1}{\lambda^{\gamma}}\frac{2^n}{\Gamma_{2^n}}\Big[\frac{\lfloor 2^n u\rfloor-\lfloor 2^n u_1\rfloor}{2^n}\Big],
\end{aligned}
$$

where C_1 is a constant depending only on a and b. Similarly, it holds

$$
\mathbb{P}\Big(\Big|L_C^{(n)}(u_2)-L_C^{(n)}(u)\Big|\geq\lambda\ \Big|\ \Gamma_{2^n}\Big)\leq\frac{C_2}{\lambda^{\gamma}}\frac{2^n}{\Gamma_{2^n}}\Big[\frac{\lfloor 2^n u_2\rfloor-\lfloor 2^n u\rfloor}{2^n}\Big], \tag{20}
$$

where C_2 is a constant depending only on a and b. From (19), we find

$$
\begin{aligned}
&\mathbb{P}\Big(\Big|L_C^{(n)}(u)-L_C^{(n)}(u_1)\Big|\geq\lambda,\ \Big|L_C^{(n)}(u_2)-L_C^{(n)}(u)\Big|\geq\lambda\Big) \\
\leq\ &\frac{C_1C_2}{\lambda^{2\gamma}}\Big[\frac{\lfloor 2^n u\rfloor-\lfloor 2^n u_1\rfloor}{2^n}\Big]\Big[\frac{\lfloor 2^n u_2\rfloor-\lfloor 2^n u\rfloor}{2^n}\Big]\mathbb{E}\Big[\Big(\frac{2^n}{\Gamma_{2^n}}\Big)^2\Big] \\
\leq\ &\frac{C}{\lambda^{2\gamma}}\Big[u_2-u_1\Big]^2,
\end{aligned}
$$

where C is a constant depending only on a and b. This completes the proof of (18). $\square$

Remark 2.1. *Let us comment on Theorem 2.1.*

1. *We can define the independent-increments MsLM* $\{L_I(x): \ x \in \mathbb{R}\}$ *on the whole line as follows. Let* $\alpha(x),\ x \in \mathbb{R}$, *be a continuous function ranging in* $[a, b] \subset (0, 2]$, *and satisfies condition (7) uniformly for all* x *in finite interval as* $t \searrow 0$. *Set the functions* $\alpha_k(x) = \alpha(x+k)$ *for all* $k \geq 0$ *and* $x \in [0, 1]$. *For any* $\alpha_k(x)$, *by Theorem 2.1, we can construct MsLM*
$$L_{I_k}(x) : [0,1] \to \mathbb{R}, \qquad k \geq 0.$$
Taking a sequence of independent processes $L_{I_k}(x),\ x \in [0,1]$, *we define* $\{L_I(x) : x \geq 0\}$ *by gluing together the parts, more precisely by*
$$L_I(x) = L_{I_{\lfloor x \rfloor}}(x - \lfloor x \rfloor) + \sum_{k=0}^{\lfloor x \rfloor - 1} L_{I_k}(1), \qquad \text{for all } x \geq 0. \tag{21}$$
Similarly, for $x < 0$, *we can define* $L_I(x) = L_I(-x)$, *since the function* $\beta(x) = \alpha(-x)$ *is defined on* $[0, +\infty)$.
2. *Let* $(\phi(n))_{n\in\mathbb{N}}$ *be a sequence of numbers satisfying* $\phi(n) \to \infty$ *as* $n \to \infty$. *Assume that* $\alpha(u)$ *is continuous in* $[0, 1]$. *By an argument similar to the proof of Theorem 2.1, the sequence of processes*
$$\widehat{L}_I^{(n)}(u) = \sum_{k=1}^{\lfloor \phi(n)u \rfloor} \left(\frac{1}{\phi(n)}\right)^{1/\alpha(\frac{k}{\phi(n)})} X(k, n)\ , \qquad u \in [0, 1], \tag{22}$$
tends in distribution to L_I *in* $(D[0, 1], d_S)$. *Since* $\alpha(u)$ *is continuous, it is easy to see that*
$$\alpha\left(\frac{\lfloor \phi(n)u \rfloor}{\phi(n)}\right) \to \alpha(u) \qquad \text{as } n \to \infty.$$
By the fact that the summands of (22) verify
$$\left(\frac{1}{\phi(n)}\right)^{1/\alpha\left(\frac{\lfloor \phi(n)u \rfloor}{\phi(n)}\right)} X(\lfloor \phi(n)u \rfloor, n) \sim S_{\alpha(\frac{\lfloor \phi(n)u \rfloor}{\phi(n)})}\left(\left(\frac{1}{\phi(n)}\right)^{1/\alpha\left(\frac{\lfloor \phi(n)u \rfloor}{\phi(n)}\right)}, 0, 0\right),$$
equality (22) means that the increment at the point u *of an* $\alpha(u)-$*multistable process* $L_I(u)$ *behaves locally like an* $\alpha(u)-$*stable random variable, but with the stability index* $\alpha(u)$ *varying with* u.
3. *If* $\alpha(u) \equiv \alpha$ *for a constant* $\alpha \in (0, 2]$, *then* $L_I(u)$ *is just the usual symmetric* $\alpha-$*stable Lévy motion* $L_\alpha(u)$. *Hence, inequality (22) gives an equivalent definition of the symmetric* $\alpha-$*stable Lévy motions: there is a sequence of independent and identically distributed*

(i.i.d.) symmetric $\alpha-$stable random variables $(Y_k)_{k\in\mathbb{N}}$ with an unit scale parameter such that

$$L_\alpha^{(n)}(u) = \sum_{k=1}^{\lfloor nu \rfloor} \frac{1}{n^{1/\alpha}} Y_k \ , \qquad u \in [0,1], \tag{23}$$

tends in distribution to L_α in $(D[0,1], d_S)$. This result is known as stable functional central limit theorem.

4. *A slightly different method to construct $L_I(u)$ can be stated as follows. Assume that $(X(\frac{k}{2^n}))_{n\in\mathbb{N},\ k=1,\ldots,2^n}$ is a family of independent and symmetric $\alpha(\frac{k}{2^n})-$stable random variables with the unit scale parameter. Then it holds*

$$L_I(u) = \lim_{n\to\infty} \sum_{k=1}^{\lfloor 2^n u \rfloor} \Big(\frac{1}{2^n}\Big)^{1/\alpha(\frac{k}{2^n})} X\Big(\frac{k}{2^n}\Big), \qquad u \in [0,1], \tag{24}$$

where convergence is in $(D[0,1], d_S)$. To highlight the differences between the two methods (8) and (24), note that $X(\frac{k}{2^n}) = X(\frac{2k}{2^{n+1}})$, while $X(k,n)$ and $X(2k,n+1)$ are two i.i.d. random variables.

5. *Inspecting the construction of field based MsLM in Falconer and Lévy Véhel [7], it seems that the sequence of processes*

$$L_F^{(n)}(u) = \sum_{k=1}^{\lfloor 2^n u \rfloor} \Big(\frac{1}{2^n}\Big)^{1/\alpha_n(u)} X(k,n), \qquad u \in [0,1], \tag{25}$$

tends in distribution to $L_F(u)$ in $(D[0,1], d_S)$. Unfortunately, it is not true in general. We have the following counter example.

Example 1. *Consider the case of $\alpha_n(u) = \alpha(u) = \frac{b}{2}\mathbf{1}_{\{0\le u\le \frac{b}{2}\}} + u\mathbf{1}_{\{\frac{b}{2}<u\le 1\}}$. The characteristic function of $L_F^{(n)}(u)$ is given by the following equality: for any $\theta \in \mathbb{R}$,*

$$\begin{aligned}
\mathbb{E}e^{i\theta L_F^{(n)}(u)} &= \prod_{k=1}^{\lfloor 2^n u \rfloor} \mathbb{E}\exp\Big\{ i\theta\Big(\frac{1}{2^n}\Big)^{1/\alpha(u)} X(k,n)\Big\} \\
&= \exp\Bigg\{ -\sum_{k=1}^{\lfloor 2^n u \rfloor} |\theta|^{\alpha(\frac{k}{2^n})}\Big(\frac{1}{2^n}\Big)^{\alpha(\frac{k}{2^n})/\alpha(u)} \Bigg\}, \qquad u \in [0,1].
\end{aligned} \tag{26}$$

Since, for all $u \in (\frac{b}{2}, 1]$ and $\theta \neq 0$,

$$\sum_{k=1}^{\lfloor 2^n u \rfloor} |\theta|^{\alpha(\frac{k}{2^n})}\Big(\frac{1}{2^n}\Big)^{\alpha(\frac{k}{2^n})/\alpha(u)} \ge \sum_{k=1}^{\lfloor 2^n b/2 \rfloor} |\theta|^{b/2}\Big(\frac{1}{2^n}\Big)^{b/2u} \to \infty, \qquad n\to\infty, \tag{27}$$

we have $L_F^{(n)}(u) \to 0$ for all $u \in (\frac{b}{2}, 1]$. Thus $L_F^{(n)}(u)$ does not tend in distribution to $L_F(u)$ in $(D[0,1], d_S)$.

3. Continuous Approximation of MsLM

It is easy to see that when $\alpha(u)$ is a constant, then the independent-increments MsLM reduce to $\alpha-$stable Lévy motions. It is well known that $\alpha-$stable Lévy motions are stochastic Hölder continuous but not continuous. We wonder if there exists a continuous approximation of independent increments MsLM? The answer is yes.

3.1. A continuous stable process

First, we shall construct a continuous stable process. To this end, we shall make use of the following useful theorem.

Theorem 3.1. *If the i.i.d. random variables $(Z_{jk})_{j,k}$ follow an $\alpha-$stable law, then it holds, for all $c > 1/\alpha$,*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty}\bigcap_{j\geq i}^{\infty} \max_{k=0,\dots,2^j-1} |Z_{jk}| \leq 2^{jc}\right) = 1.$$

Proof. We only need to show that, for all $c > 1/\alpha$,

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty}\bigcup_{j\geq i}^{\infty} \max_{k=0,\dots,2^j-1} |Z_{jk}| > 2^{jc}\right) = 0.$$

By Borel-Cantelli Lemma, it is sufficient to prove, for all $c > 1/\alpha$,

$$\sum_{j\geq 1} \mathbb{P}\left(\max_{k=0,\dots,2^j-1} |Z_{jk}| > 2^{jc}\right) < \infty. \tag{28}$$

To prove (28), we need the following technical lemma (cf. Property 1.2.15 of Samorodnitsky and Taqqu [13] for details).

Lemma 3.1. *Let $Z \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$. Then*

$$\begin{cases} \lim_{\lambda\to\infty} \lambda^\alpha \mathbb{P}(Z > \lambda) & = & C_\alpha \frac{1+\beta}{2}\sigma^\alpha, \\ \lim_{\lambda\to\infty} \lambda^\alpha \mathbb{P}(Z < -\lambda) & = & C_\alpha \frac{1-\beta}{2}\sigma^\alpha. \end{cases}$$

Return to the proof of (28). For all $c > 1/\alpha$ and all j large enough, we have

$$\begin{aligned} \mathbb{P}\left(\max_{k=0,\dots,2^j-1} |Z_{jk}| > 2^{jc}\right) &= 1 - \mathbb{P}\left(|Z_{jk}| \leq 2^{jc} \text{ for all } k = 0, \dots, 2^j - 1\right) \\ &= 1 - \prod_{k=0}^{2^j-1} \mathbb{P}\left(|Z_{jk}| \leq 2^{jc}\right). \end{aligned} \tag{29}$$

Then, by equality (29) and Lemma 3.1, we deduce

$$\begin{aligned}\mathbb{P}\left(\max_{k=0,\dots,2^j-1}|Z_{jk}|>2^{jc}\right) &= 1-\left(1+O\Big(\frac{1}{2^{j\alpha c}}\Big)\right)^{2^j}\\ &= O\Big(\frac{1}{2^{j(\alpha c-1)}}\Big), \quad j\to\infty.\end{aligned}$$

Thus we obtain (28) for all $c>1/\alpha$. $\square$

In the following theorem, we give a construction of continuous stable process. First, we recall the definition of the "triangle" function:

$$\varphi(t)=\begin{cases}2t & \text{for } t\in[0,1/2)\\ 2-2t & \text{for } t\in[1/2,1]\\ 0 & \text{otherwise.}\end{cases}$$

Define $\varphi_{jk}(t)=\varphi(2^jt-k)$, for $j=0,1,\dots$, and $k=0,\dots,2^j-1$.

Theorem 3.2. *Assume the i.i.d. random variables $(Z_{jk})_{j,k}$ follow a symmetric $\alpha-$stable law with the unit scale parameter. Then, for all $d>1/\alpha$, the process*

$$X(t)=\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}2^{-jd}Z_{jk}\varphi_{jk}(t), \qquad t\in[0,1],$$

is a continuous and symmetric α-stable process. When $d=1/\alpha$, the process $X(t)$ is also a symmetric, may not be continuous, α-stable process in $L^p(\Omega\times[0,1])$ for any $0<p<\alpha$.

Proof. Set $X_{-1}\equiv 0$ and define the sequence of processes $(X_j)_{j\in\mathbb{N}}$ by:

$$X_j(t)=X_{j-1}(t)+\sum_{k=0}^{2^j-1}2^{-jd}Z_{jk}\varphi_{jk}(t).$$

First we show that the sequence of processes $(X_j)_{j\in\mathbb{N}}$ converges almost surely uniformly. Indeed, for all t,

$$X_j(t)-X_{j-1}(t)=\sum_{k=0}^{2^j-1}2^{-jd}Z_{jk}\varphi_{jk}(t).$$

Since the functions $(\varphi_{jk})_{j,k}$ have disjoint supports and $|\varphi_{jk}|\le 1$, it follows that

$$||X_j(t)-X_{j-1}(t)||_\infty=2^{-jd}\max_{k=0,\dots,2^{j-1}}|Z_{jk}|.$$

Theorem 3.1 entails that $(X_j)_{j\in\mathbb{N}}$ converges almost surely in $C([0,1],||\cdot||_\infty)$ to a continuous process X for all $d>1/\alpha$. When $d=1/\alpha$, we show that the sequence $(X_j)_{j\in\mathbb{N}}$ converges to a

random variable X in $L^p(\Omega \times [0,1])$ for any $0 < p < \alpha$. Indeed, for any $0 < p < \alpha$,

$$\begin{aligned}
\int_0^1 \mathbb{E}|X_j(t) - X_{j-1}(t)|^p dt &\leq 2^{-jp/\alpha}\mathbb{E}|Z_{00}|^p \sum_{k=0}^{2^j-1} \int_0^1 \varphi_{jk}^p(t)dt \\
&\leq 2^{-jp/\alpha}\mathbb{E}|Z_{00}|^p \int_0^1 \varphi_{00}^p(t)dt \\
&= 2^{-jp/\alpha}\mathbb{E}|Z_{00}|^p, \qquad (30)
\end{aligned}$$

this entitles convergence of $(X_j)_{j\in\mathbb{N}}$.

Next, we prove that X is a symmetric $\alpha-$stable process. By Theorem 3.1.2 of Samorodnitsky and Taqqu (1994), we only need to check that all linear combinations

$$\sum_{k=1}^d b_k X(t_k), \quad d \geq 1, \quad t_1, ..., t_d \in [0,1] \text{ and } b_1, ..., b_d \text{ real}$$

are symmetric $\alpha-$stable. We distinguish two cases as follows. Define

$$D_n = \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\}$$

and $D = \bigcup_{n=0,1,...} D_n$.

i) If $t_k \in D$, then all random variables $X(t_k), 1 \leq k \leq d$, are symmetric $\alpha-$stable. Thus all linear combinations $\sum_{k=1}^d b_k X(t_k)$ are symmetric and $\alpha-$stable.

ii) For $t_k \in [0,1], 1 \leq k \leq d$, we have $t_{kl} \in D$ such that $t_{kl} \to t_k, l \to \infty$. Since X is continuous, we have

$$\sum_{k=1}^d b_k X(t_k) = \lim_{j\to\infty} \sum_{k=1}^d b_k X(t_{kl}).$$

Its characteristic function has the following form:

$$\mathbb{E}\exp\left\{ i\theta \sum_{k=1}^d b_k X(t_k) \right\} \quad = \quad \lim_{l\to\infty} \mathbb{E}\exp\left\{ i\theta \sum_{k=1}^d b_k X(t_{kl}) \right\}.$$

It is easy to see that the scale parameter of $\sum_{k=1}^d b_k X(t_{kl})$ is

$$\sigma_l(\alpha) = \left(\sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} \Big(\sum_{k=1}^d |b_k| 2^{-jd} \varphi(2^j t_{kl} - i) \Big)^{\alpha} \right)^{1/\alpha}.$$

Since at most one summand of the sum $\sum_{i=0}^{2^j-1} 2^{-jd}\varphi(2^j t - i)$ is non-zero and

$$\sum_{i=0}^{2^j-1} \Big(\sum_{k=1}^d |b_k| 2^{-jd} \varphi(2^j t_{kl} - i) \Big)^{\alpha} \leq d\, b^{\alpha} 2^{-j\alpha d},$$

where $b = \max\{|b_k|, 1 \le k \le d\}$, then $\sigma(\alpha) = \lim_{l\to\infty} \sigma_l(\alpha)$ exists for $d \ge 1/\alpha$ and

$$\begin{aligned}\mathbb{E}\exp\Big\{i\theta\sum_{k=1}^{d} b_k X(t_k)\Big\} &= \lim_{j\to\infty}\exp\Big\{-\sigma_l(\alpha)^\alpha|\theta|^\alpha\Big\}\\ &= \exp\Big\{-\sigma(\alpha)^\alpha|\theta|^\alpha\Big\}.\end{aligned}$$

This implies that all linear combinations $\sum_{k=1}^{d} b_k X(t_k)$ are symmetric $\alpha-$stable random variables. This completes the proof. □

One deduces the scale parameter $\sigma(t)$ of the process $X(t)$ is given as follows

$$\sigma^\alpha(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\Big(2^{-jd}\varphi(2^j t-k)\Big)^\alpha.$$

By noting that at most one $\varphi(2^j t-k)$ is non-zero for all j, we have the following estimation of the scale parameter

$$\varphi^{1/\alpha}(t) \ \le\ \sigma(t) \ \le\ \Big(\frac{1}{1-2^{-\alpha d}}\Big)^{1/\alpha}, \qquad t\in[0,1].$$

It is worth noting that when $t \ne 0, 1$, we have $\sigma(t) > 0$. This observation will be useful to establish continuous approximations of MsLM in the next subsection.

3.2. Continuous approximations of MsLM

In Theorem 2.1, we establish discrete approximations of the independent-increments MsLM. In this subsection, we shall give continuous approximations of the independent-increments MsLM. It is worth to noting that one cannot make use of the method of Theorem 3.2 to establish continuous approximations of MsLM in general, since a sum of two stable random variables with different stability indices is not a stable random variable. To obtaining continuous approximations of the independent-increments MsLM, our main method is to replace the summands in (8) by a sequence of independent and continuous stable processes starting at 0, for instance the stable processes established in Theorem 3.2.

Theorem 3.3. *Let $\alpha(u)$ be a continuous function ranging in $[a,b] \subset (0,2]$. Assume that $\Big(X_{\alpha(\frac{k}{2^n})}(t)\Big)_{n\in\mathbb{N},\,k=0,\ldots,2^n-1}$ is a family of independent and continuous $\alpha(\frac{k}{2^n})-$stable random processes. Assume $X_{\alpha(\frac{k}{2^n})}(0) = 0$ and $\sigma_{\alpha(\frac{k}{2^n})}(t) > 0$ for all $t\in(0,1]$ and all $n\in\mathbb{N}$, $k = 0,...,2^n-1$, where $\sigma_{\alpha(\frac{k}{2^n})}(t)$ is the scale parameter of $X_{\alpha(\frac{k}{2^n})}(t)$. Define*

$$\begin{aligned}S_n(u) \;=\; &\Big(\frac{1}{2^n}\Big)^{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}\frac{1}{\sigma_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}(\frac{1}{2^n})}X_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}\Big(u-\frac{\lfloor 2^n u\rfloor}{2^n}\Big)\\ &+\sum_{k=0}^{\lfloor 2^n u\rfloor-1}\Big(\frac{1}{2^n}\Big)^{\alpha(\frac{k}{2^n})}\frac{1}{\sigma_{\alpha(\frac{k}{2^n})}(\frac{1}{2^n})}X_{\alpha(\frac{k}{2^n})}\Big(\frac{1}{2^n}\Big), \qquad u\in[0,1]. \qquad (31)\end{aligned}$$

Then $(S_n)_{n\in\mathbb{N}}$ is a sequence of continuous processes and the process $S_n(u), u \in [0,1]$, tends in distribution to $L_I(u)$ in $(D[0,1], d_S)$.

By the definition of $S_n(u)$ in (31), it seems that the process $S_n(u)$ restores more and more details of $L_I(u)$ when n is increasing.

It is worth noting that when $\alpha(u) \equiv \alpha$ for a constant $\alpha \in (0,2]$, Theorem 3.3 gives continuous approximations to the usual symmetric $\alpha-$stable Lévy motion $L_\alpha(u)$.

Proof. It is easy to see that the first item in the right hand side of (31) converges to zero in distribution as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \left(\left(\frac{1}{2^n}\right)^{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})} \frac{1}{\sigma_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}(\frac{1}{2^n})} X_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}\left(u - \frac{\lfloor 2^n u\rfloor}{2^n}\right)\right) = 0 \tag{32}$$

in distribution. Notice that the summands

$$\frac{1}{\sigma_{\alpha(\frac{k}{2^n})}(\frac{1}{2^n})} X_{\alpha(\frac{k}{2^n})}\left(\frac{1}{2^n}\right) \tag{33}$$

in the right hand side of (31) are independent $\alpha(\frac{k}{2^n})-$stable random variables with the unit scale parameter. Using Theorem 2.1, we find that the process $S_n(u), u \in [0,1]$, tends in distribution to $L_I(u)$ in $(D[0,1], d_S)$. $\square$

4. Integrals of Multistable Lévy Measure

Let $\alpha = \alpha(u), u \in [0,1]$, be a càdlàg function ranging in $[a,b] \subset (0,2]$. Denote by

$$\mathcal{L}_\alpha[0,1] = \Big\{ f : f \text{ is measurable with } ||f||_\alpha < \infty \Big\},$$

where

$$||f||_\alpha := \inf\left\{ \lambda > 0 : \int_0^1 \Big|\frac{f(x)}{\lambda}\Big|^{\alpha(x)} dx = 1 \right\} \qquad \text{and} \qquad ||0||_\alpha = 0.$$

Note that $||\cdot||_\alpha$ is a quasinorm; see Falconer and Liu [8] and Ayache [1]. Using the Kolmogorov consistency conditions and the Lévy continuity theorem, Falconer and Liu [8] (see also Falconer [5]) proved that the characteristic function, for all $(\theta_1, ..., \theta_d) \in \mathbb{R}^d$,

$$\mathbb{E}\exp\left\{ i\Big(\sum_{j=1}^d \theta_j I(f_j)\Big)\right\} = \exp\left\{ -\int \Big|\sum_{j=1}^d \theta_j f_j(x)\Big|^{\alpha(x)} dx \right\} \tag{34}$$

well defines a consistent probability distribution of the random vector $(I(f_1), I(f_2), ..., I(f_d)) \in \mathbb{R}^d$ on the functions $f_j \in \mathcal{L}_\alpha[0,1]$, where $I(f) = \int f(x) M_\alpha(dx)$. They called M_α the multistable Lévy measure and $I(f) = \int f(x) M_\alpha(dx)$ the integral with respect to M_α. Moreover, they also

showed that the integrals of functions with disjoint supports are independent. In particular, it holds

$$L_I(u) = \int \mathbf{1}_{[0,\ u]}(x) M_\alpha(dx), \qquad u \in [0,1].$$

In the following theorem, we give an alternative definition of the integrals based on the weighted sums of independent random variables.

Theorem 4.1. *Let $(X(k,n))_{n\in\mathbb{N},\ k=1,\ldots,2^n}$ be defined by Theorem 2.1. Then, for any $f \in \mathcal{L}_\alpha[0,1]$, it holds*

$$\int_0^1 f(x)M_\alpha(dx) = \lim_{n\to\infty} \sum_{k=1}^{2^n} \Big(\frac{1}{2^n}\Big)^{1/\alpha(\frac{k}{2^n})} f\Big(\frac{k}{2^n}\Big) X(k,n) \tag{35}$$

in distribution.

Proof. Denote by

$$S(k,n) = \Big(\frac{1}{2^n}\Big)^{1/\alpha(\frac{k}{2^n})} f\Big(\frac{k}{2^n}\Big) X(k,n) \quad \text{and} \quad X_n = \sum_{k=1}^{2^n} S(k,n).$$

It is easy to see that, for any $\theta \in \mathbb{R}$,

$$\mathbb{E}e^{i\,\theta X_n} \;=\; \prod_{k=1}^{2^n} \mathbb{E}e^{i\theta S(k,n)} = \exp\Big\{ - \Big|\theta f\Big(\frac{k}{2^n}\Big)\Big|^{\alpha(k/2^n)} \frac{1}{2^n}\Big\}.$$

Hence, we have

$$\lim_{n\to\infty} \mathbb{E}e^{i\,\theta X_n} \;=\; \exp\Big\{ - \int_0^1 \Big|\theta f(x)\Big|^{\alpha(x)} dx\Big\},$$

which means $\lim_{n\to\infty} X_n = \int_0^1 f(x)M_\alpha(dx)$ in distribution by the definition of the multistable integrals with respect to the multistable Lévy measure M_α. □

The following theorem relates the convergence of a sequence of $\alpha(u)-$multistable integrals to the convergence of the sequence of integrands.

Theorem 4.2. *Assume $X_j = \int_0^1 f_j(x)M_\alpha(dx)$ and $X = \int_0^1 f(x)M_\alpha(dx)$, for $f_j, j = 1,2,..., f \in \mathcal{L}_\alpha[0,1]$. Then*

$$\lim_{j\to\infty} X_j = X$$

in probability, or

$$\lim_{j\to\infty} (X_j - X) = 0$$

in distribution, if and only if

$$\lim_{j\to\infty} ||f_j - f||_\alpha = 0.$$

Proof. The convergence $\lim_{j\to\infty} X_j = X$ in probability is equivalent to $\lim_{j\to\infty}(X_j - X) = 0$ in probability and hence to the convergence in distribution to zero of the sequence $(X_j - X)_{j=1,2,...}$. If $X_j - X$ convergence in distribution to 0, then, for any $\theta \in \mathbb{R}$,

$$1 = \lim_{j\to\infty} \mathbb{E}e^{i\,\theta(X_j - X)} = \lim_{j\to\infty} \exp\left\{ -\int_0^1 \Big|\theta\Big(f_j(x) - f(x)\Big)\Big|^{\alpha(x)} dx \right\}, \tag{36}$$

which is equivalent to, for any $\lambda > 0$,

$$\lim_{j\to\infty} \int_0^1 \Big|\frac{f_j(x) - f(x)}{\lambda}\Big|^{\alpha(x)} dx = 0.$$

This equality means $\lim_{j\to\infty} ||f_j - f||_\alpha = 0$. □

The last theorem shows that convergence in probability of multistable integrals coincides with convergence in quasinorm $||\cdot||_\alpha$.

The convergence $\lim_{j\to\infty} X_j = X$ almost surely implies the convergence $\lim_{j\to\infty} X_j = X$ in probability. Thus the following corollary is obvious.

Corollary 4.1. *Assume that $X_j, j = 1, 2, ..$ and X are defined by Theorem 4.2. If*

$$\lim_{j\to\infty} X_j = X$$

almost surely, then

$$\lim_{j\to\infty} ||f_j - f||_\alpha = 0.$$

4.1. Independence

Independence of two multistable integrals imposes a stronger restriction on the integrands: they must almost surely have disjoint supports with respect to Lebesgue measure $\mathcal{L}$. Indeed,

Theorem 4.3. *Let $X_1 = \int_0^1 f_1(x)M_\alpha(dx)$ and $X_2 = \int_0^1 f_2(x)M_\alpha(dx)$ be two multistable integrals, where $f_j \in \mathcal{L}_\alpha[0, 1], j = 1, 2$. Assume either*

$$[a, b] \subset (0, 2)$$

or

$$f_1(x)f_2(x) \geq 0 \qquad \mathcal{L} - a.s.\ on\ [0,\ 1]. \tag{37}$$

Then X_1 and X_2 are independent if and only if

$$f_1(x)f_2(x) \equiv 0 \qquad \mathcal{L} - a.s.\ on\ [0,\ 1]. \tag{38}$$

Proof. Two multistable integrals X_1 and X_2 are independent if and only if, for any $(\theta_1, \theta_2) \in \mathbb{R}^2$,

$$\mathbb{E} \exp\Big\{ i(\theta_1 X_1 + \theta_2 X_2) \Big\} \quad = \quad \mathbb{E} \exp\Big\{ i\theta_1 X_1 \Big\} \, \mathbb{E} \exp\Big\{ i\theta_2 X_2 \Big\}. \tag{39}$$

Notice that

$$\mathbb{E} \exp\Big\{ i(\theta_1 X_1 + \theta_2 X_2) \Big\} = \exp\Bigg\{ -\int_0^1 \Big| \sum_{j=1}^{2} \theta_j f_j(x) \Big|^{\alpha(x)} dx \Bigg\},$$

and that

$$\mathbb{E} \exp\Big\{ i\theta_1 X_1 \Big\} \, \mathbb{E} \exp\Big\{ i\theta_2 X_2 \Big\} = \exp\Bigg\{ -\sum_{j=1}^{2} \int_0^1 \Big| \theta_j f_j(x) \Big|^{\alpha(x)} dx \Bigg\}.$$

Equating the moduli of (39) gives

$$\int_0^1 \Big| \sum_{j=1}^{2} \theta_j f_j(x) \Big|^{\alpha(x)} dx = \sum_{j=1}^{2} \int_0^1 \Big| \theta_j f_j(x) \Big|^{\alpha(x)} dx. \tag{40}$$

Notice that (40) implies that

$$\int_0^1 \Big| f_1(x) - f_2(x) \Big|^{\alpha(x)} dx \quad = \quad \int_0^1 \Big| f_1(x) \Big|^{\alpha(x)} dx + \int_0^1 \Big| f_2(x) \Big|^{\alpha(x)} dx. \tag{41}$$

$$= \quad \int_0^1 \Big| f_1(x) + f_2(x) \Big|^{\alpha(x)} dx \tag{42}$$

Assume $[a, b] \subset (0, 2)$. We argue as Lemma 2.7.14 of Samorodnitsky and Taqqu [13]. When $\alpha \in (0, 2)$, the function $r_\alpha(u) = u^{\alpha/2}, u \geq 0$, is strictly concave. Therefore, for fix $x \in [0, 1]$,

$$\begin{aligned} &|f_1(x) + f_2(x)|^{\alpha(x)} + |f_1(x) - f_2(x)|^{\alpha(x)} \\ &\qquad = 2 \, \frac{r_{\alpha(x)}(|f_1(x) + f_2(x)|^2) + r_{\alpha(x)}(|f_1(x) - f_2(x)|^2)}{2} \\ &\qquad \leq 2 \, r_{\alpha(x)}\Big(\frac{|f_1(x) + f_2(x)|^2 + |f_1(x) - f_2(x)|^2}{2} \Big) \\ &\qquad = 2 \, r_{\alpha(x)}(f_1(x)^2 + f_2(x)^2) \\ &\qquad \leq 2 \, (|f_1(x)|^{\alpha(x)} + |f_2(x)|^{\alpha(x)}) \end{aligned} \tag{43}$$

with equality in the preceding relations equivalent $f_1(x) f_2(x) = 0$. Inequalities (41) and (42) imply that

$$\begin{aligned} &\int_0^1 \Big| f_1(x) - f_2(x) \Big|^{\alpha(x)} dx + \int_0^1 \Big| f_1(x) + f_2(x) \Big|^{\alpha(x)} dx \\ &\qquad\qquad = 2 \Big(\int_0^1 \Big| f_1(x) \Big|^{\alpha(x)} dx + \int_0^1 \Big| f_2(x) \Big|^{\alpha(x)} dx \Big). \end{aligned} \tag{44}$$

Now (43) implies that the left-hand side of the last equality is always less than or equal to the right-hand side of the last inequality and, if they are equal, then necessarily (38) holds.

Assume (37). Then it holds $|f_1(x) - f_2(x)| \leq |f_1(x) + f_2(x)|$ $\mathcal{L} - a.s.$ on $[0, 1]$. When $\alpha \in (0, 2]$, the function $r_\alpha(u) = u^{\alpha/2}$ is increasing in $u \in [0, \infty)$. Hence (42) holds if and only if (38) holds.

This proves that (38) is a necessary condition for the independence of X_1 and X_2. It is also sufficient because if (38) holds, then (40) also holds. $\square$

The preceding result is very useful and will often be used in the sequel.

Theorem 4.4. *Assume $f_j \in \mathcal{L}_\alpha[0, 1]$, $j = 1, ..., d$. Assume either $[a, b] \subset (0, 2)$ or $f_i(x)f_k(x) \geq 0$ $\mathcal{L}$−a.s. on $[0, 1]$ for any subset $\{i, k\}$ of $\{1, 2, ..., d\}$. The multistable integrals $X_j = \int_0^1 f_j(x)M_\alpha(dx)$, $j = 1, ..., d$, are independent if and only if they are pairwise independent, i.e., if and only if*

$$f_i(x)f_k(x) \equiv 0 \quad \mathcal{L} - a.s. \text{ on } [0, 1] \tag{45}$$

for any subset $\{i, k\}$ of $\{1, 2, ..., d\}$.

Proof. Independence clearly implies pairwise independence. By Theorem 4.3, pairwise independence implies (45). If (45) holds, then it holds, for any $(\theta_1, ..., \theta_d) \in \mathbb{R}^d$,

$$\int_0^1 \Big|\sum_{j=1}^d \theta_j f_j(x)\Big|^{\alpha(x)} dx = \sum_{j=1}^d \int_0^1 \Big|\theta_j f_j(x)\Big|^{\alpha(x)} dx. \tag{46}$$

Thus the joint characteristic function of $X_1, ..., X_d$ factorizes

$$\begin{aligned} \mathbb{E} \exp\Big\{i\sum_{j=1}^d \theta_j X_j\Big\} &= \exp\Bigg\{-\int_0^1 \Big|\sum_{j=1}^d \theta_j f_j(x)\Big|^{\alpha(x)} dx\Bigg\} \\ &= \exp\Bigg\{-\sum_{j=1}^d \int_0^1 \Big|\theta_j f_j(x)\Big|^{\alpha(x)} dx\Bigg\} \\ &= \prod_{j=1}^d \mathbb{E}\exp\Big\{i\theta_j X_j\Big\}. \end{aligned}$$

This proves that $X_1, ..., X_d$ are independent. $\square$

4.2. Stochastic Hölder continuity

We call a random process $X(u)$, $u \in I$, is *stochastic Hölder continuous* of exponent $\beta \in (0, 1]$, if it holds

$$\limsup_{u,r\in I,\ |u-r|\to 0} \mathbb{P}(|X(u) - X(r)| \geq C|u - r|^\beta) = 0$$

for a positive constant C. It is obvious that if $X(u)$ is stochastic Hölder continuous of exponent $\beta_1 \in (0, 1]$, then $X(u)$ is stochastic Hölder continuous of exponent $\beta_2 \in (0, \beta_1]$.

Example 2. *Assume that a random process $X(u)$, $u \in I$, satisfies the following condition: there exist three strictly positive constants γ, c, ρ such that*

$$\mathbb{E}|X(u) - X(r)|^{\gamma} \leq c\,|u - r|^{\rho}, \qquad u, r \in I.$$

Then $X(u), u \in I$, is stochastic Hölder continuous of exponent $\beta \in (0, \min\{1, \rho/\gamma\})$. Indeed, it is easy to see that for all $u, r \in I$,

$$\begin{aligned}\mathbb{P}\Big(|X(u) - X(r)| \geq C|u - r|^{\beta}\Big) &\leq \frac{\mathbb{E}|X(u) - X(r)|^{\gamma}}{C^{\gamma}|u - r|^{\beta\gamma}} \\ &\leq \frac{c}{C^{\gamma}}|u - r|^{\rho - \beta\gamma},\end{aligned}$$

which implies our claim.

The following theorem gives a sufficient condition such that the integrals with respect to multistable Lévy measure M_α are stochastic Hölder continuous.

Theorem 4.5. *Assume that $X(t) = \int_0^1 f(t, x) M_\alpha(dx)$ is a multistable integral, where $f(t, x)$ is jointly measurable and $f(t, x) \in \mathcal{L}_\alpha[0, 1]$ for all $t \in I$. If there exist two constants $\eta > 0$ and $C > 0$ such that*

$$\int_0^1 \Big|f(t, s) - f(v, s)\Big|^{\alpha(s)} ds \ \leq\ C\Big|t - v\Big|^{\eta}, \qquad t, v \in I. \tag{47}$$

Then it holds

$$\mathbb{P}(|X(t) - X(v)| \geq |t - v|^{\beta}) \leq C_{a,b}\ |t - v|^{\eta - b\beta}, \qquad t, v \in I, \tag{48}$$

where $C_{a,b}$ is a constant depending on a, b and C. In particular, it implies that $X(t)$ is stochastic Hölder continuous of exponent $\beta \in (0, \min\{1, \eta/b\})$.

Proof. By the Billingsley inequality (cf. p. 47 of [2]) and (47), it is easy to see that, for all $t, v \in I$ and all $x > 0$,

$$\begin{aligned}\mathbb{P}(|X(t) - X(v)| \geq x) &\leq \frac{x}{2}\int_{-2/x}^{2/x}\Big(1 - \mathbb{E}e^{i\theta\left(X(t) - X(v)\right)}\Big)\, d\theta \\ &= \frac{x}{2}\int_{-2/x}^{2/x}\left(1 - \exp\left\{-\int_0^1 \Big|\theta\Big(f(t, z) - f(v, z)\Big)\Big|^{\alpha(z)} dz\right\}\right) d\theta \\ &\leq \frac{x}{2}\int_{-2/x}^{2/x}\int_0^1 \Big|\theta\Big(f(t, z) - f(v, z)\Big)\Big|^{\alpha(z)} dz\, d\theta \\ &\leq \frac{x}{2}\left[\int_{|\theta|<1}\Big|\theta\Big|^{a} d\theta + \int_{1\leq|\theta|\leq 2/x}\Big|\theta\Big|^{b} d\theta\right] C\Big|t - v\Big|^{\eta} \\ &\leq C\left(\frac{x}{a+1} + \frac{2^{b+1}}{b+1}\frac{1}{x^{b}}\right)\Big|t - v\Big|^{\eta}.\end{aligned}$$

Taking $x = |t-v|^{\beta}$, we obtain (48). This implies that $X(t)$ is stochastic Hölder continuous of exponent $\beta \in (0, \min\{1, \eta/b\})$. $\square$

As an example to illustrate Theorem 4.5, consider the weighted MsLM introduced by Falconer and Liu [8]. The following theorem shows that the weighted MsLM are Hölder continuous of exponent $\beta \in (0, \min\{1, 1/b\})$.

Theorem 4.6. *Let*

$$Y(t) = \int_0^1 w(x) \mathbf{1}_{[0,\ t]}(x) M_\alpha(dx), \quad t \in [0,1],$$

be a weighted multistable Lévy motion, where the function $w(x), x \in [0,1]$, is càdlàg. Then $Y(t)$ is stochastic Hölder continuous of exponent $\beta \in (0, \min\{1, 1/b\})$. Moreover, it holds

$$\mathbb{P}(|Y(t) - Y(v)| \geq |t-v|^{\beta}) \leq C_{a,b}\ |t-v|^{1-b\beta}, \qquad t, v \in [0,1], \tag{49}$$

where $C_{a,b}$ is a constant depending on $a, b, \alpha(\cdot)$ and $w(\cdot)$. In particular, it implies that $L_I(u), u \in [0,1]$, is stochastic Hölder continuous of exponent $\beta \in (0, \min\{1, 1/b\})$.

Proof. Set $f(t,x) = w(x)\mathbf{1}_{[0,t]}(x),\ t, x \in [0,1]$. It is easy to see that, for all $v, t \in [0,1]$ such that $v \leq t$,

$$\begin{aligned} \int_0^1 \Big| f(t,s) - f(v,s) \Big|^{\alpha(s)} ds &\leq \int_0^1 \Big| w(s) \mathbf{1}_{[v,\ t]}(s) \Big|^{\alpha(s)} ds \\ &\leq C_\omega \int_0^1 \mathbf{1}_{[v,\ t]}(s)\, ds \\ &\leq C_\omega\, (t-v), \end{aligned}$$

where $C_\omega = \sup_{z \in [0,1]} |w(z)|^{\alpha(z)}$. By Theorem 4.5, we get (49). This completes the proof of Theorem 4.6. $\square$

4.3. Strongly localisability

When the function $\alpha(x) \in [a,b], x \in [0,1]$, is continuous, some sufficient conditions such that the multistable integrals are localisable (or strongly localisable) has been obtained by Falconer and Liu. In the following theorem, we give some new conditions such that localisability can be strengthened to strongly localisability.

Theorem 4.7. *Assume that $f(t,x)$ and $h(t,x)$ are jointly measurable; and that $f(t,x), h(t,x) \in \mathcal{L}_\alpha[0,1]$ for any $t \in [0,1]$. Assume that $X(t) = \int_0^1 f(t,x) M_\alpha(dx)$ and $X_x'(t) = \int_0^1 h(t,x) M_\alpha(dx)$ are two multistable integrals and have versions in $D[0,1]$. Suppose that $X(t)$ is $1/\alpha(x)-$localisable at x with local form $X_x'(t)$. If there exist two constants $\eta > 1$ and $C > 0$ such that*

$$\int_0^1 \left| \frac{f(x+rt,s) - f(x+rv,s)}{r^{1/\alpha(x)}} \right|^{\alpha(s)} ds \ \leq\ C\Big|t-v\Big|^{\eta}, \qquad t, v \in [0,1], \tag{50}$$

for all sufficiently small $r > 0$, then $X(t)$ is strongly localisable at all $x \in [0,1]$. Moreover, if $X(t)$ has independent increments and (50) holds for a constant $\eta > 1/2$, then the claim holds also.

Notice that condition (50) is slightly more general than the condition of Falconer and Liu (cf. Theorem 3.2 of [8]): there exist two constants $\eta > 1/a$ and $C > 0$ such that

$$\left\| \frac{f(x+rt,\cdot) - f(x+rv,\cdot)}{r^{1/\alpha(x)}} \right\|_\alpha \;\le\; C\Big|t-v\Big|^\eta, \qquad t, v \in [0,1], \tag{51}$$

for all sufficiently small $r > 0$.
Proof. For any $x \in [0,\,1)$, define

$$X_r(u) = \frac{X(x+ru) - X(x)}{r^{1/\alpha(x)}}, \qquad r, u \in (0,\,1].$$

By Theorem 15.6 of Billingsley [2], it suffices to show that, for some $\beta > 1$ and $\tau \ge 0$,

$$\mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \ge \lambda,\ \Big|X_r(u_2) - X_r(u)\Big| \ge \lambda\Big) \le \frac{C}{\lambda^\tau}\Big[u_2 - u_1\Big]^\beta \tag{52}$$

for $u_1 \le u \le u_2, \lambda > 0$ and $r \in (0,1]$, where C is a positive constant. Since $X_r(u) - X_r(u_1)$ and $X_r(u_2) - X_r(u)$ are symmetric, it follows that

$$\begin{aligned}
&\mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \ge \lambda,\ \Big|X_r(u_2) - X_r(u)\Big| \ge \lambda\Big) \\
\le\;& 4\,\mathbb{P}\Big(X_r(u) - X_r(u_1) + \Big(X_r(u_2) - X_r(u)\Big) \ge 2\lambda\Big) \\
=\;& 4\,\mathbb{P}\Big(X_r(u_2) - X_r(u_1) \ge 2\lambda\Big).
\end{aligned}$$

By the Billingsley inequality (cf. p. 47 of [2]) and (50), we have

$$\begin{aligned}
\mathbb{P}\Big(X_r(u_2) - X_r(u_1) \ge 2\lambda\Big) \;&\le\; \lambda \int_{-1/\lambda}^{1/\lambda} \Big(1 - \mathbb{E}e^{i\theta\left(X_r(u_2) - X_r(u_1)\right)}\Big)\,d\theta \\
&=\; \lambda \int_{-1/\lambda}^{1/\lambda} \Big(1 - e^{-\int_0^1 \left|\theta \frac{f(x+ru_2,s) - f(x+ru_1,s)}{r^{1/\alpha(x)}}\right|^{\alpha(s)} ds}\Big)\,d\theta \\
&\le\; \lambda \int_{-1/\lambda}^{1/\lambda} \int_0^1 \left|\theta \frac{f(x+ru_2,s) - f(x+ru_1,s)}{r^{1/\alpha(x)}}\right|^{\alpha(s)} ds\,d\theta \\
&=\; \lambda \int_{-1/\lambda}^{1/\lambda} |\theta|^\mu \int_0^1 \left|\frac{f(x+ru_2,s) - f(x+ru_1,s)}{r^{1/\alpha(x)}}\right|^{\alpha(s)} ds\,d\theta \\
&\le\; \frac{C_1}{\lambda^\gamma}\Big[u_2 - u_1\Big]^\eta,
\end{aligned} \tag{53}$$

where $\mu = a\mathbf{1}_{[1,\,\infty)}(\theta) + b\mathbf{1}_{(0,\,1)}(\theta), \gamma = a\mathbf{1}_{[1,\,\infty)}(\lambda) + b\mathbf{1}_{(0,\,1)}(\lambda)$ and C_1 is a positive constant depending only on a, b and C. Thus

$$\mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \ge \lambda,\ \Big|X_r(u_2) - X_r(u)\Big| \ge \lambda\Big) \;\le\; \frac{4C_1}{\lambda^\gamma}\Big[u_2 - u_1\Big]^\eta.$$

Hence, by (52), if $\eta > 1$, then $X(t)$ is $1/\alpha(x)-$strongly localisable at x with strong local form $X'_x(t)$.

If $X(t)$ has independent increments, then

$$\begin{aligned}\mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \geq \lambda,\ \Big|X_r(u_2) - X_r(u)\Big| \geq \lambda\Big) \\ = \ \mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \geq \lambda\Big)\,\mathbb{P}\Big(\Big|X_r(u_2) - X_r(u)\Big| \geq \lambda\Big). \end{aligned} \tag{54}$$

By an argument similar to (53), it follows that

$$\mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \geq \lambda\Big) \ \leq \ \frac{4C_1}{\lambda^\gamma}\Big[u - u_1\Big]^\eta$$

and

$$\mathbb{P}\Big(\Big|X_r(u_2) - X_r(u)\Big| \geq \lambda\Big) \ \leq \ \frac{4C_1}{\lambda^\gamma}\Big[u_2 - u\Big]^\eta.$$

Using the inequality $xy \leq (x+y)^2/4,\ x, y \geq 0$, we have

$$\mathbb{P}\Big(\Big|X_r(u) - X_r(u_1)\Big| \geq \lambda,\ \Big|X_r(u_2) - X_r(u)\Big| \geq \lambda\Big) \leq \frac{16C_1^2}{\lambda^{2\gamma}}\Big[u_2 - u_1\Big]^{2\eta}. \tag{55}$$

Thus, if $2\eta > 1$, by (52), then $X(t)$ is $1/\alpha(x)-$strongly localisable at x. This completes the proof of the theorem. □

As an example to illustrate Theorem 4.7, consider the weighted MsLM. Falconer and Liu have proved that the weighted MsLM are localisable. The following theorem shows that the weighted MsLM are not only localisable but also strongly localisable. In particular, it shows that the independent-increments MsLM are strongly localisable.

Theorem 4.8. *Assume that the function $\alpha(u), u \in [0,1]$, satisfies condition (7). Let*

$$Y(t) = \int_0^1 w(x)\mathbf{1}_{[0,\ t]}(x) M_\alpha(dx), \qquad t \in [0,1],$$

be a weighted multistable Lévy motion, where the function $w(x), x \in [0,1]$, is continuous. Then $Y(t)$ is $1/\alpha(x)-$strongly localisable at all $x \in [0,1]$ with strong local form $w(x)L_{\alpha(x)}(\cdot)$. In particular, this implies that $L_I(t)$ is $1/\alpha(x)-$strongly localisable at all $x \in [0,1]$ with strong local form $L_{\alpha(x)}(\cdot)$, an $\alpha(x)-$stable Lévy motion.

Proof. It is known that $Y(t)$ is $1/\alpha(x)-$localisable at all x with strong local form $w(x)L_{\alpha(x)}(\cdot)$; see Falconer and Liu [8]. Set $f(t,x) = w(x)\mathbf{1}_{[0,\ t]}(x),\ t, x \in [0,1]$. By (16), the integrand of

$Y(t)$ satisfies, for all $t, v \in [0,1]$ such that $v \le t$,

$$\begin{aligned}
\int_0^1 \left| \frac{f(x+rt,s) - f(x+rv,s)}{r^{1/\alpha(x)}} \right|^{\alpha(s)} ds &= \int_0^1 \left| \frac{w(s)\mathbf{1}_{[x+rv,\, x+rt]}(s)}{r^{1/\alpha(x)}} \right|^{\alpha(s)} ds \\
&= \int \left| w(x+rz)\mathbf{1}_{[v,\ t]}(z) \right|^{\alpha(x+rz)} r^{(\alpha(x)-\alpha(x+rz))/\alpha(x)} dz \\
&\le C_w \int \mathbf{1}_{[v,\ t]}(z)\, r^{(\alpha(x)-\alpha(x+rz))/\alpha(x)} dz \\
&\le 2C_w\ (t-v),
\end{aligned}$$

for all sufficiently small $r > 0$, where $s = x + rz$ and $C_\omega = \sup_{z\in[0,1]} |w(z)|^{\alpha(z)}$. By the fact that the integrals of functions with disjoint supports are independent, it is easy to see that $Y(t)$ has independent increments, the first claim of the theorem follows by Theorem 4.7. In particular, since $L_I(t) = \int_0^1 \mathbf{1}_{[0,\ t]}(x) M_\alpha(dx)$, the first claim of the theorem implies the second one with $w(x) = 1, x \in [0,1]$. □

Remark 4.1. *By inspecting the proof of Falconer and Liu [8], we can see that $Y(t)$ is also $1/\alpha(x)-$localisable at all x with strong local form $w(x)L_{\alpha(x)}(t)$ when the function $w(x), x \in [0,1]$, is càdlàg. Hence, Theorem 4.8 holds true when the function $w(x), x \in [0,1]$, is càdlàg.*

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